COMBINATORIAL TANGENT SPACE AND RATIONAL SMOOTHNESS OF SCHUBERT VARIETIES

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Résumé

A la suite de Contou Carrere [CC], nous considérons la résolution de Bott-Samelson d'une variété de Schubert comme une variété de galeries de l'immeuble de Tits associé à la situation. Nous montrons que la lissité rationnelle d'une telle variété est codée par un sous-espace de l'espace tangent de Zariski appelé, l'espace tangent combinatoire. Nous utilisons pour cela une caractérisation de la lissité rationelle d'une variété de Schubert introduite par Carrell et Peterson [CP].

Abstract

Following Contou Carrere [CC], we consider the Bott-Samelson resolution of a Schubert variety as a variety of galleries in the Tits building associated to the situation. We prove that the rational smoothness of a Schubert variety can be expressed in terms of a subspace of the Zariski tangent space called, the combinatorial tangent space. For this, we use a characterization of rational smoothness of a Schubert variety introduced by Carrell and Peterson [CP].

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1 Introduction

In this paper, we introduce a combinatorial tool in the study of the singular locus of a Schubert variety, called the combinatorial tangent space. This space is built from a Bott-Samelson resolution of such a variety and is extracted from the work of Contou Carrere [CC]. Using a combinatorial result of Carrell and Peterson [CP, Theorem C] which characterizes, via the Kashdan-Lusztig polynomials, the rational smoothness of a Schubert variety, we show our main result (Section 4, Theorem 4):

Theorem. *The Schubert variety $\overline{\Sigma}(B, w)$ is rationally smooth at a $T-$fixed point $x = u(P)$ if and only if for all $u \leq v \leq w$, the dimension of the combinatorial tangent space of $\overline{\Sigma}(B, w)$ at $v(P)$ is equal to the dimension of the variety.*

Here is an outline of the paper, in §2, we state some definitions and some combinatorial settings. In particular, we recall from [CC], the universal Schubert scheme and the construction of the Bott-Samelson variety as a variety of galleries in the Tits building associated to any semi-simple (adjoint) group G. In §3, we give the definition of the combinatorial tangent space and state some of these properties. In §4, we relate explicitly the combinatorial tangent space and the rational smoothness of a Schubert variety. And we show how this space is connected to the set of $T-$invariant curves in such a variety.

Althought the construction of the combinatorial tangent space of a Schubert variety is presented here in the theory of semi-simple groups, this construction is still valid in the more general setting of Kac-Moody groups and their Schubert varieties [G]. Indeed, in this context, Bott-Samelson and Schubert varieties can be defined [Ma] and they have a combinatorial interpretation, of course, by taking into account the building associated to such a group. The notion of "épinglage" (cf. Section 2 below) does not exist for a Kac-Moody group G, but we still have a basis (infinite this time) of the tangent space of G/B at any $T-$fixed point (given by the presentation of the Kac-Moody algebra). Moreover, the characterization of rational smoothness we use generalizes to this case ([KL] and [C1]). So does our main result : the rational smoothness of the Schubert varieties is characterized by the combinatorial tangent space.

This approach, i.e. studying the smoothness of a Schubert variety via the Bott-Samelson resolution (viewed as variety of galleries), follows the work of Contou Carrere [CC]. And it has many others applications, some of them could be found in my Ph. D. thesis [G] that I am working out at Université Monpellier 2 (France).

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2 Settings and Definitions

In this section, we use partially the notations of [SGA3] and we present the Schubert variety, following Contou Carrere [CC], in a non usual way, which is quite near from the point of view of Kazhdan and Lusztig [KL].

Let $G = (G, T, M, R)$ be a semi-simple (adjoint) split group over an algebraically closed field k. That is, T is a maximal torus of G, M is an abelian group such that $Hom_{k-gr}\big(T, \mathbb{G}_m(k)\big) \simeq M_k$ and R is a root system of (G, T). We have the following decomposition of $\mathcal{G} = Lie(G)$ under the $T-$adjoint action :

$$\mathcal{G} = \mathcal{T} \oplus \bigoplus_{\alpha \in R} \mathcal{G}^{\alpha},$$

where the $\mathcal{G}^{\alpha}$'s are one dimensional vector spaces.

Let $\varepsilon = (R_0, \{X_\alpha\}_{\alpha \in R_0})$ be an "épinglage" of G. That is, we choose a set R_0 of simple roots in R (this choice corresponds to the choice of a Borel subgroup B of G such that $T \subset B \subset G$) and for each $\alpha \in R_0$, we choose X_α a generator of the vector space $\mathcal{G}^{\alpha}$.

Let $W = Norm_G(T)/T$ denote the Weyl group of (G, T) and let S be a set of generators of W adapted to the choice of $B \subset G$ (in particular, there is a bijection between R_0 and S, we thus index the elements of S by simple roots). Moreover, for a part t of S, $W_t \subset W$ is the subgroup of W generated by $\{s_\alpha\}_{\alpha \in t}$.

2.1 Universal Schubert Scheme

Following [CC, Part I, §4], we introduce the universal Schubert scheme, we refer to this work for more details and a full description of these notions.

For any set E, $\mathcal{P}(E)$ stands for the set of all his parts. Let $Par(G)$ denote the $k-$variety of all the parabolic subgroups of G. Herewith the "épinglage" of G, we get a decomposition of this variety:

$$Par(G) = \coprod_{t \in \mathcal{P}(R_0)} Par_t(G),$$

where $Par_t(G)$ is defined as the variety of parabolic subgroups of type t. The variety of Borel subgroups of G is then $Bor(G) = Par_\emptyset(G)$.

Let Σ denote the variety $Par(G) \times_k Par(G)$. The group G acts diagonally on Σ and the quotient is denoted by $t.st = \Sigma/G$. This is the constant $k-$scheme :

$$\coprod_{t, t' \in \mathcal{P}(R_0)} W_t \backslash W / W_{t'}.$$

In the sequel of what we wrote, we can define an immersion :

$$\Sigma \hookrightarrow Par(G) \times_k t.st \times_k Par(G),$$

and state the following definition.

Definition 1. *We call Σ the universal Bruhat cell and $\overline{\Sigma}$, the scheme theoretic image of Σ in $Par(G) \times_k t.st \times_k Par(G)$ by this immersion, is called the universal Schubert scheme.*

Remark 1. *If we fix a parabolic subgroup P of type t_P such that $P \supset B$, where B is the Borel subgroup of G given by the "épinglage", and an element $\overline{w} \in W/W_{t_P}$, then by taking fibres in the previous definition, we recover the Bruhat cell $\Sigma(B, \overline{w})$ and the Schubert variety $\overline{\Sigma}(B, \overline{w})$:*

$$\overline{\Sigma}(B, \overline{w}) \hookrightarrow \{B\} \times \{\overline{w}\} \times Par_{t_P}(G) \simeq G/P.$$

The cell $\Sigma(B, \overline{w})$ is the variety of parabolic subgroups of type t_P of G in relative position $\overline{w}$ with B and $\overline{\Sigma}(B, \overline{w})$ is the variety of parabolic subgroups of type t_P of G in relative position $\overline{u}$ with B for $\overline{u} \leq \overline{w}$ in W/W_{t_P} (here $\leq$ represent the relative Chevalley-Bruhat order).

In the following, we will only consider Schubert varieties in $Par_{t_P}(G) \simeq G/P$, where $P \supset B$ is a parabolic subgroup of G of type t_P.

2.2 Tits Building

Now, let us go into a succint description of a combinatorial object naturally associated to a semi-simple group, the Tits building (actually, this construction is valid without any hypothesis on the ground field k).

We denote by $\Delta(G)$ the set of all parabolic subgroups of G ordered by the opposite relation of the inclusion between parabolic subgroups. This operation endowes this set with a structure of simplicial complex. The variety $Par(G)$ and the complex $\Delta(G)$ have the same underlying set, but different structures.

As we have fixed a maximal torus, $T \subset G$, we can define a subcomplex $\mathbb{A}$ associated to T, called an apartment, as follows :

$$\mathbb{A} = \{P \in Par(G),\ T \subset P\},$$

if $P \in \mathbb{A}$ then there exists a parabolic subgroup Q containing B and $n \in N = Norm_G(T)$ such that $P = nQn^{-1}$.

And the building can be expressed as $\Delta(G) = G \times \mathbb{A}/\sim$ where $\sim$ is the equivalence relation :

$$(x, P) \sim (x', P') \Leftrightarrow \begin{cases} P = P' \\ x^{-1}x' \in P. \end{cases}$$

For each maximal torus in G there is an apartment corresponding to it and in order to recover $\Delta(G)$ we paste all these subcomplexes with respect to the equivalence relation $\sim$. This description of $\Delta(G)$ as a "union" of apartments provides a better handing to deal with the building.

Elements of $\Delta(G)$ are called faces, to each parabolic subgroup P of G is associated a face, denoted by F_P. A chamber of $\Delta(G)$ is a maximal face. The chambers are associated to the Borel subgroups of G.

The group W acts on the faces of $\mathbb{A}$ and in a simply transitively way on the set of all the chambers of $\mathbb{A}$. Moreover, we can define a distance between two chambers of $\mathbb{A}$:

$$d(C, C') = l(w_{CC'}),$$

where $w_{CC'}$ is the element of W which maps C to C' and $l(w)$, for $w \in W$ is the length of any reduced decomposition of w.

To each face F_P contained in the chamber F_B, we associate its type :

$$typ(F_P) = t \subset S,$$

where t is defined by $W_t = Stab_W(F_P)$. Therefore, we get $typ(F_B) = \emptyset$ since $Stab_W(F_B) = \{1\} = W_\emptyset$. The action of W allows us to define the type of each face of $\mathbb{A}$. Moreover, thanks to the retraction of $\Delta(G)$ onto $\mathbb{A}$ with centre F_B [T, 3,3, p.42], we can define the type of all the faces (as there is a bijection between R_0 and S, the two notions of type mentionned are the same).

The set of types, denoted by $typ\big(\Delta(G)\big)$ is still a complex and $typ : \Delta(G) \to typ\big(\Delta(G)\big)$ is a morphism of complex.

A *folding* of $\mathbb{A}$ is an idempotent and type preserving morphism of complex $\phi : \mathbb{A} \to \mathbb{A}$ such that each chamber C belonging to $\phi(\mathbb{A})$ is the image of exactly two chambers of which itself [T, 1,8, p.7]. The image of a folding $\phi(\mathbb{A})$ is called a *root* of $\mathbb{A}$.

If α is a *root* of $\mathbb{A}$, we define M_α, the *wall associated to* α, as the subcomplex of $\mathbb{A}$ composed by the faces F such that there are two adjacent chambers C and C' (i.e. $C \cap C'$ is a codimension 1 face of C and of C') with $C \in \alpha, C' \notin \alpha$ and $F \subset C \cap C'$.

If we look at a graphic representation of $\mathbb{A}$ as the root system of (G, T) (for instance see [BOU, Planche X]), the *roots* correspond to the roots of (G, T) and the walls correspond to the hyperplans related to the reflections. To each wall M_β in $\mathbb{A}$, we can associate two foldings ϕ_β and $\phi_{-\beta}$, corresponding to the roots β and $-\beta$.

Moreover, we say that a folding ϕ is *towards a chamber* C if C belongs to the image of ϕ. For example, the foldings towards F_B are indexed by the positive roots.

2.3 Generalized Galleries

The notion of gallery that we are using is a bit more general than the one presented in Tits' book [T]. So, in order to make a difference, we will use the expression $g-$gallery to denote such a gallery.

Definition 2 ([CC, Part I, §2]). *Let $(E, <)$ be an ordered set. A sequel $(e_n, ..., e_0)$ of elements of E is a $g-$gallery if those elements fit into one of the following situations :*

$$(e_n > e_{n-1} < \cdots > e_1 < e_0)$$
$$(e_n < e_{n-1} > \cdots > e_1 < e_0)$$
$$(e_n > e_{n-1} < \cdots < e_1 > e_0)$$
$$(e_n < e_{n-1} > \cdots < e_1 > e_0).$$

For example, let g be a $g-$gallery of $\Delta(G)$ of which faces verify the following incidence relations :

$$g = (F_r \supset F'_r \subset F_{r-1} \supset \cdots \supset F'_1 \subset F_0 \supset F'_0).$$

We define the type of g to be the $g-$gallery of types τ :

$$\tau = typ(g) = (t_r \subset t'_r \supset t_{r-1} \subset \cdots \subset t'_1 \supset t_0 \subset t'_0),$$

where each t_i (resp. t'_j) is the type of the corresponding face.

Actually, g corresponds to a configuration of parabolic subgroups verifying the following inclusions :

$$g = (Q_r \subset P_r \supset Q_{r-1} \subset \cdots \subset P_1 \supset Q_0 \subset P_0),$$

where the Q_j's are of type t_j and the P_i's, of type t'_i.

Definition 3. *If F_r and F'_0 are two faces of $\mathbb{A}$ (resp. of $\Delta(G)$), we denote by*

$$\Gamma(\mathbb{A}, \tau, F_r, F'_0) \quad (resp.\ \Gamma(\Delta(G), \tau, F_r, F'_0))$$

the finite set (resp. the set) of all $g-$galleries contained in $\mathbb{A}$ (resp. in $\Delta(G)$), of type τ, of source F_r and of target F'_0.

Now, we present the notion of minimality for $g-$galleries.

For two faces F and F' of $\mathbb{A}$ such that $F' \subset F$, we denote by $\mathcal{M}_{F'}(F)$ the set of walls M such that $F' \in M, F \notin M$.

Definition 4 ([CC, Part I, §5]). *A g-gallery $\gamma = (F_r \supset F'_r \subset F_{r-1} \supset \cdots \supset F'_1 \subset F_0 \supset F'_0) \in \Gamma(\mathbb{A}, \tau, F_r, F'_0)$, between the chamber $F_r = F_B$ and the face F'_0 is said to be minimal if :*

i) for $r \geq i \geq 1$ the sets $\mathcal{M}_{F'_i}(F_{i-1})$ are disjoint ;

ii) $\mathcal{M}(F_B, F'_0) = \cup_{r \geq i \geq 1} \mathcal{M}_{F'_i}(F_{i-1})$, where $\mathcal{M}(F_B, F'_0)$ denote the set of walls which separate F_B from F'_0.

For a discussion on further properties of minimal $g-$galleries, we refer to [CC].

2.4 Chamber Galleries

Definition 5. *A g-gallery* $g = (F_r \supset F'_r \subset F_{r-1} \supset \cdots \supset F'_1 \subset F_0 \supset F'_0)$ *is called a chamber g-gallery if*

i) for $r \geq i \geq 0$, F_i *is a chamber ;*

ii) for $r \geq j \geq 1$, F'_j *is a face of codimension 1 in* F_j *and* F_{j-1}.

The chamber $g-$galleries are nearly galleries in the sense of Tits [T], except that we allow their target face to be of any type (instead of being a chamber). By the way, we will make the abuse of language and call our $g-$galleries, simply galleries.

So, the minimality of a chamber gallery g is expressed in the same way as in [T]. Keeping the notations as in definition 5, g is minimal if :

i) for $r \geq i \geq 1$, F_i and F_{i-1} are adjacent chambers inside $\mathbb{A}$;

ii) $\#\mathcal{M}(F_B, F'_0) = r$.

Now, if we fix $\tau = (t_r \subset t'_r \supset t_{r-1} \subset \cdots \subset t'_1 \supset t_0 \subset t'_0)$ a minimal chamber gallery type, then $t_i = typ(B) = \emptyset$ for $r \geq i \geq 0$ and $t'_j = \{s_{i_j}\}$ where $s_{i_j} \in S$ is a generator of W and t'_0 is any type. The word $s_{i_r} \cdots s_{i_1}$ built with these reflections, is a reduced decomposition of an element of W.

More precisely, we have the following

Proposition 1. *i) We keep the notations and* τ *is fixed as above. If* F_r *and* F_0 *are two chambers of* $\mathbb{A}$ *such that* $d(F_r, F_0) = r \geq 0$ *and* F'_0 *is the face of type* t'_0 *contained in* F_0*, then* $\Gamma(\mathbb{A}, \tau, F_r, F'_0) = \{\gamma_w\}$ *is reduced to a single element.*

ii) For any type t_P *(*t_P *is the type of a parabolic subgroup* $P \supset B$*), the construction above states a bijection between minimal chamber gallery types and reduced decompositions of minimal length coset representatives of the classes in* W/W_{t_P}.

We define an operation on the galleries of $\Gamma(\mathbb{A}, \tau, F_r, -)$ by applying to all the faces of a gallery a folding towards F_r. Therefore, all these galleries can be obtained from γ_w, by a finite number of foldings towards F_r.

2.5 Configuration Variety

Now, we use the combinatorial setting of the building and of the galleries to describe the Bott-Samelson variety.

Let $P \supset B$ be a parabolic subgroup of G and let $\overline{w} \in W/W_{t_P}$. Also, let w denote the element of minimal length in $\overline{w}$, and let $w = s_{i_r} \cdots s_{i_1}$ be a reduced decomposition of w, denoted by $\underline{i} = (i_r, ..., i_1)$.

In the rest of this paper, w will always mean this element.

We fix $F'_0 = F_{\overline{w}(P)}$, where $\overline{w}(P)$ stands for $\overline{w}P\overline{w}^{-1}$. Let

$$\tau_{\underline{i}} = (t_r \subset t'_r \supset t_{r-1} \subset \cdots \subset t'_1 \supset t_0 \subset t'_0)$$

be the type of chamber gallery in $\mathbb{A}$ between F_B and $F_{\overline{w}(P)}$ associated to the reduced decomposition $\underline{i}$, that is $t'_j = \{s_{i_j}\}$, for $r \geq j \geq 1$ and $t'_0 = t_P$. The type $\tau_{\underline{i}}$ is therefore a minimal chamber gallery type.

Definition-Proposition 1 ([CC, Part I, §6]). *The configuration variety $CONF_{\tau_{\underline{i}}}(G)_B$ is defined as the subvariety of the product*

$$\Pi(\tau_{\underline{i}}) = \prod_{r \geq i \geq 0} Par_{t_i}(G) \times Par_{t'_i}(G)$$

which consists of all the configurations of parabolic subgroups of the form :

$$g = (B_r \subset P_r \supset B_{r-1} \subset \cdots \subset P_1 \supset B_0 \subset P_0),$$

where $B_r = B$ and $typ(g) = \tau_{\underline{i}}$.

As we saw it before, these configurations are the chamber galleries in $\Delta(G)$, of type $\tau_{\underline{i}}$ and of source F_B.

To each gallery $g \in \Gamma(\Delta(G), \tau_{\underline{i}}, F_B, -)$, we associate its target which is a face of type $t'_0 = t_P$, associated to a parabolic subgroup in relative position $\overline{u}$ with B, for $\overline{u} \leq \overline{w}$ in W/W_{t_P}. Thus, we obtain a morphism of variety :

$$\pi : CONF_{\tau_{\underline{i}}}(G)_B \rightarrow \overline{\Sigma}(B, \overline{w}).$$

Beside, from the decomposition $\underline{i} = (i_r, \cdots i_1)$, we can construct the Bott-Samelson variety $\hat{\Sigma}(B, \tau_{\underline{i}}) = P_{i_r} \times_B \cdots \times_B (P_{i_1}/B)$, where for $r \geq j \geq 1$, $P_{i_j} = Bs_{i_j}B \cup B$ is the unique parabolic subgroup of type t'_j containing B (see for example [De, §3, Definition 1]). We denote by $[x_r, ..., x_1]$ the $k-$points of this variety.

Proposition 2 ([CC, Part I, §6]). *The morphism*

$$f : \hat{\Sigma}(B, \tau_{\underline{i}}) \rightarrow CONF_{\tau_{\underline{i}}}(G)_B$$

defined in the following way, is an isomorphism :

$$f([x_r, \cdots, x_1]) = (B_r \subset P_r \supset B_{r-1} \subset \cdots \subset P_1 \supset B_0 \subset P_0)$$

where
$B_i = x_r \cdots x_{i+1} B (x_r \cdots x_{i+1})^{-1}$, *for* $r-1 \geq i \geq 0$,
$P_j = x_r \cdots x_{j+1} P_{i_j} (x_r \cdots x_{j+1})^{-1}$, *for* $r-1 \geq j \geq 1$,
$B_r = B$ *and* $P_0 = x_r \cdots x_1 P (x_r \cdots x_1)^{-1}$.

Thus, we arrive to the well-known

Theorem 1 ([Ha], [De], [CC, Part I, §6]). *The couple $\big(CONF_{\tau_{\underline{i}}}(G)_B, \pi\big)$ is a smooth equivariant resolution of the Schubert variety $\overline{\Sigma}(B, \overline{w})$, called the Bott-Samelson resolution.*

In particular, π is a birational proper morphism.

Remark 2. *Demazure [De] and Hansen [Ha] have first shown independently that the Bott-Samelson variety is a smooth resolution of the schubert variety, but Contou Carrere [CC] enunciates this theorem in the more general setting of the universal Schubert scheme over any base scheme.*

3 Combinatorial Tangent Space

From now on, we will denote the configuration variety as the Bott-Samelson variety by $\hat{\Sigma}(B, \tau_{\underline{i}})$. And we will call the galleries of $\Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, -)$, *combinatorial galleries*. Furthermore, we will use freely the two writings of galleries i.e. $\gamma = [\gamma_r, ..., \gamma_1] = (F_B = F_r \supset F'_r \subset F_{r-1} \supset \cdots \supset F'_1 \subset F_0 \supset F'_0)$. Let us remark that $\gamma = [\gamma_r, ..., \gamma_1]$ is a combinatorial gallery iff $\gamma_j = s_{i_j}$ or $\gamma_j = 1$.

3.1 The tangent space at γ of $\hat{\Sigma}(B, \tau_{\underline{i}})$

Let $Q \supset T$ be a parabolic subgroup of G. As G is "épinglé", we know that $Lie(Q) = \mathcal{T} \oplus \big(\bigoplus_{\alpha \in R_Q} \mathcal{G}^\alpha\big)$, where R_Q is the set of roots of (Q, T) and where $\mathcal{G}^\alpha =< X_\alpha >$.

If x denote the $k-$point of $Par_{t_Q}(G)$ given by Q, then

$$T_x Par_{t_Q}(G) \simeq Lie(G)/Lie(Q) = \bigoplus_{\alpha \in R \setminus R_Q} \mathcal{G}^\alpha.$$

Let $\tau_{\underline{i}} = (t_r \subset t'_r \supset t_{r-1} \subset \cdots \subset t'_1 \supset t_0 \subset t'_0)$ be the type of a minimal chamber gallery given by a fixed reduced decomposition of $w \in \overline{w}$. Let also $\gamma = (F_{B_r} \supset F_{P_r} \subset F_{B_{r-1}} \supset \cdots \supset F_{P_1} \subset F_{B_0} \supset F_{P_0})$ denote a gallery of $\Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, F_{\overline{u}(P)})$, where $\overline{u} \leq \overline{w}$ in W/W_{t_P} $(P_0 = \overline{u}(P) = \overline{u} P \overline{u}^{-1})$.

From the definition-proposition 1, we deduce that $T_\gamma \hat{\Sigma}(B, \tau_{\underline{i}})$ is a vector subspace of the product :

$$\mathbb{P}(\gamma) = \big(\mathcal{G}/Lie(B_r) \times \mathcal{G}/Lie(P_r)\big) \times \cdots \times \big(\mathcal{G}/Lie(B_0) \times \mathcal{G}/Lie(P_0)\big).$$

Now, we explain, following [CC, Part II, §5] how to construct a basis of this vector space.

Let $E(\tau_{\underline{i}}) = \{t_r, t'_r, t_{r-1}, \cdots, t'_1, t_0, t'_0\}$ be the set of types which appears in $\tau_{\underline{i}}$ totally ordered by the reading order (i.e. $t_r \geq t'_r \geq t_{r-1} \cdots$). Let

$$\mathbb{E}(\tau_{\underline{i}}) = \coprod_{\theta \in E(\tau_{\underline{i}})} \{\theta\} \times (R \setminus R_\theta) \quad \text{where} \quad R_\theta = \begin{cases} R_{B_i} & \text{if} \quad \theta = t_i \\ R_{P_j} & \text{if} \quad \theta = t'_j. \end{cases}$$

We define an equivalence relation $\sim$ on $\mathbb{E}(\tau_{\underline{i}})$:

$$(\theta, \alpha) \sim (\theta', \alpha') \Leftrightarrow \begin{cases} \alpha = \alpha' \\ \alpha \in (R \setminus R_\theta) \cap (R \setminus R_{\theta-1}) \cap \cdots \cap (R \setminus R_{\theta'}) \quad (\theta \geq \theta'). \end{cases}$$

Define $\mathbb{B}(\gamma) = \mathbb{E}(\gamma)/\sim$ and $\mathbb{B}(\gamma, B) = \{C \in \mathbb{B}(\gamma), C \cap \big(\{t_r\} \times (R \setminus R_B)\big) = \emptyset\}$. For $C \in \mathbb{B}(\gamma, B)$, denote X_C the element of $\mathbb{P}(\gamma)$ with $\theta-$component equals to X_α if $(\theta, \alpha) \in C$ and 0 otherwise.

Proposition 3 ([CC, Part II, §5]). *The set of vectors $\{X_C\}_{C\in\mathbb{B}(\gamma,B)}$ is a basis of $T_\gamma\hat{\Sigma}(\tau_{\underline{i}},B)$.*

Remark 3. *For $C\in\mathbb{B}(\gamma,B)$, a vector X_C has the form :*

$$X_C=\big((0,0),...,(0,0),X_\alpha,...,X_\alpha,\underbrace{0,0,...,0}_{\text{eventually}}\big).$$

3.2 Combinatorial Tangent Space

Let x be the $k-$point of $\overline{\Sigma}(B,\overline{w})$ given by a parabolic subgroup of type t_P containing T. That is $x=\overline{u}(P)$ for $\overline{u}\le\overline{w}$ in W/W_{t_P}. Let us denote $\Gamma=\Gamma(\mathbb{A},\tau_{\underline{i}},F_B,F_x)$, the finite set of galleries in $\mathbb{A}$, of type $\tau_{\underline{i}}$, of source F_B and of target F_x. These combinatorial galleries are the $T-$fixed points of the fibre $\hat{\Sigma}(B,\tau_{\underline{i}})_x=\pi^{-1}(x)$ (we shall discuss more precisely this point in section 4).

Definition 6. *The combinatorial tangent space, $T^c_x\overline{\Sigma}(B,\overline{w})$, is the vector subspace of $\overline{\Sigma}(B,\overline{w})$ generated by the union of the images of tangent spaces $T_\gamma\hat{\Sigma}(B,\tau_{\underline{i}})$ for all $\gamma\in\Gamma$,*

$$T^c_x\overline{\Sigma}(B,\overline{w})=\Big\langle\bigcup_{\gamma\in\Gamma}T_\gamma\pi\big(T_\gamma\hat{\Sigma}(B,\tau_{\underline{i}})\big)\Big\rangle.$$

This is a vector subspace of $Lie(G)/Lie(\overline{u}(P))=\bigoplus_{\alpha\in R\setminus R_{\overline{u}(P)}}\mathcal{G}^\alpha$ and by construction we know a basis of it, composed by the X_α with $\alpha\in R\setminus R_{\overline{u}(P)}$, which comes from a X_C, such that $X_C=\big((0,0),...,(0,0),X_\alpha,...,X_\alpha\big)$.

So, $T^c_x\overline{\Sigma}(B,\overline{w})=\bigoplus_{\alpha\in\mathcal{R}_x}\mathcal{G}^\alpha$, where $\mathcal{R}_x\subset R\setminus R_{\overline{u}(P)}$ defines the combinatorial tangent space as a $T-$weighted subspace of $Lie(G)/Lie(\overline{u}(P))$.

We give here some definitions to understand how a gallery will contribute to the construction of the combinatorial tangent space.

From now on, we will only consider galleries inside $\mathbb{A}$.

Definition 7. *A bend $(F_{B'},F_{P'})$ is a gallery of the form $F_{B'}\supset F_{P'}\subset F_{B'}$, where $F_{B'}$ is a chamber of $\mathbb{A}$ and $F_{P'}$ is a codimension 1 face of $F_{B'}$ carried by a wall $M_{\beta'}$.*

Remark 4. *A generator vector X_β of $T^c_x\overline{\Sigma}(B,\overline{w})$ comes from a gallery $\gamma\in\Gamma$ and β is associated to a wall M_β of $\mathbb{A}$. Two cases arise*

1) γ crosses M_β in such a way that X_β appears in the last component of a X_C, $C\in\mathbb{B}(\gamma,B)$ (this is the case for all the walls separating F_B from F_x);

2) γ contains a bend $(F_{B'},F_{P'})$ where $F_{P'}$ is carried by M_β, again in such a way that X_β appears in the last component of an X_C, $C\in\mathbb{B}(\gamma,B)$. Such bends are called generating bends.

A chamber gallery which contains no bend is minimal, and it crosses one and only one time each wall which separates its source from its target.

As an example, let us suppose that G is the simple group of type B_2 and let us take $P = B$ the "standard" Borel subgroup and $w = s_\alpha s_\beta s_\alpha$ in the Weyl group which is the dihedral group generated by s_α and s_β. Let us denote by P_α (resp. P_β) the unique parabolic subgroup of type $\{s_\alpha\}$ (resp. $\{s_\beta\}$) containing B.

The following figure shows the graphic representation of the apartment $\mathbb{A}$ associated to the "standard" torus $T \subset B$ and of the root system of type B_2. We draw the eight combinatorial galleries of $\Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, -)$ ($\tau_{\underline{i}}$ is the type associated to the reduced decomposition $w = s_\alpha s_\beta s_\alpha$) as curves in the graphic representation of $\mathbb{A}$. For example γ_0^0 is the unique gallery of type $\tau_{\underline{i}}$ contained in the chamber F_B, so its curve stay inside the chamber while bending the walls corresponding to the faces F_{P_α} and F_{P_β}. And the curve corresponding to $\gamma_{\alpha\beta} = [s_\alpha, s_\beta, 1] = \big(F_B \supset F_{P_\alpha} \subset F_{s_\alpha(B)} \supset F_{s_\alpha(P_\beta)} \subset F_{s_\alpha s_\beta(B)} \supset F_{s_\alpha s_\beta(P_\alpha)} \subset F_{s_\alpha s_\beta(B)}\big)$ follows the faces which appear in its second writing. And so on for the other galleries. The bullet at the beginning of each curve stands for the source of the gallery and the arrow at the end shows the direction and the target.

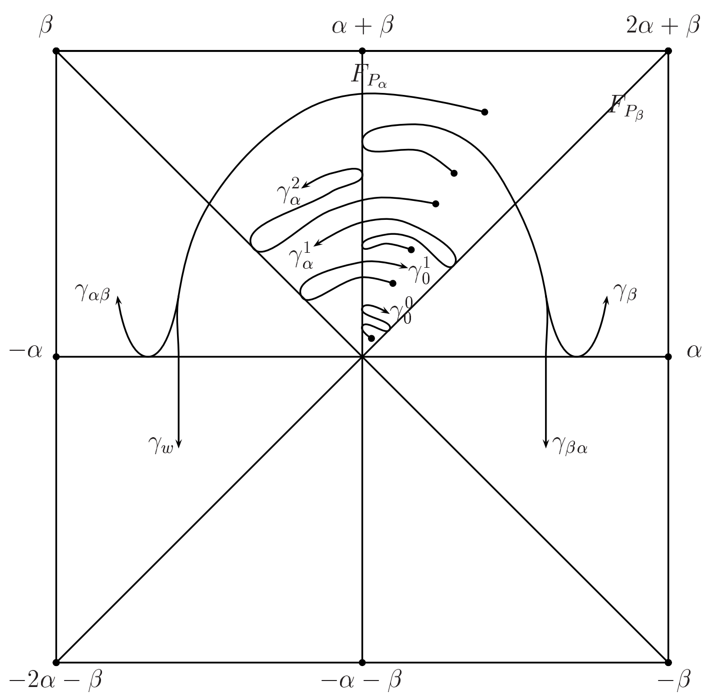

Figure 3.2. Combinatorial galleries in an apartment of type B_2, case $w = s_\alpha s_\beta s_\alpha$.

As we can see on this picture, if $x = s_\alpha B s_\alpha$, the combinatorial tangent space $T_x^c\overline{\Sigma}(B, w)$ admits the basis $\{X_{-\beta}, X_{-2\alpha-\beta}, X_\alpha\}$. And these generators are all given by generating bends of the two galleries of $\Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, F_x) = \{\gamma_\alpha^1, \gamma_\alpha^2\}$.

On the other hand, if $x = s_\alpha s_\beta(B) = s_\alpha s_\beta B(s_\alpha s_\beta)^{-1}$, then $T_x\overline{\Sigma}(B, w) = T_x^c\overline{\Sigma}(B, w) =< X_\alpha, X_{2\alpha+\beta}, X_{-\alpha-\beta} >$. This time, the two first generators are given

by the fact that $\gamma_{\alpha\beta}$ crosses the walls M_α and $M_{2\alpha+\beta}$ and the third by the generating bend $(F_{s_\alpha s_\beta(B)}, F_{s_\alpha s_\beta(P_\alpha)})$.

It seems that $T^c_x\overline{\Sigma}(B,\overline{w})$ depends on the choice of a type $\tau_{\underline{i}}$, i.e. on the choice of a reduced decomposition of w, but it is not the case.

Proposition 4. *The combinatorial tangent space $T^c_x\overline{\Sigma}(B,\overline{w})$ is independent from the choice of the reduced decomposition of w.*

Proof. Take two decompositions $\underline{i}$ and $\underline{j}$ of w. The decomposition $\underline{j}$ can be obtained from $\underline{i}$ by the action of an element of the generalized braid group. This element can be written as a finite sequence of generalized braid relations. So, it suffices to show that generators of $T^c_x\overline{\Sigma}(B,\overline{w})$ are the same if we "calculate" them with $\underline{i}$ or with a transform of $\underline{i}$ by a generalized braid relation r_t. Let $\underline{j} = r_t(\underline{i})$, the relation r_t induces a $T-$equivariant isomorphism of variety

$$f_{r_t} : \hat{\Sigma}(\tau_{\underline{i}}, B) \xrightarrow{\simeq} \hat{\Sigma}(\tau_{\underline{j}}, B),$$

(hence, $f_{r_t}\big(\Gamma(\tau_{\underline{i}})\big) = \Gamma(\tau_{\underline{j}})$). Let X_β be a generator of $T^c_x\overline{\Sigma}(B,\overline{w})$ coming from $\gamma \in \Gamma(\tau_{\underline{i}})$. Whether γ contains a minimal chamber gallery δ which crosses M_β, then $f_{r_t}(\gamma)$ will contain $f_{r_t}(\delta)$ which still crosses M_β. Whether β is given by a generating bend μ of γ, again $f_{r_t}(\mu)$ will be a generating bend of $f_{r_t}(\gamma)$ and will still give X_β as a generator of $T^c_x\overline{\Sigma}(B,\overline{w})$. So, in the two cases, proposition holds.

□

In the following, we will study more closely the set $\Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, F_x)$ of all the combinatorial galleries above F_x. This will allow us to show a pleasant property of the combinatorial tangent space. In order to do that, we need some definitions.

Definition 8. *Let $\gamma = [\gamma_r, ..., \gamma_1] = (F_B = F_r \supset F'_r \subset F_{r-1} \supset \cdots \supset F'_1 \subset F_0 \supset F'_0) \in \Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, F_x)$. A buckle of γ is a sub-gallery of γ of the shape*

$$\mathfrak{b} = (F'_j \subset F_{j-1} \supset \cdots \supset F'_{i+1} \subset F_i \supset F'_i),$$

where $r \geq j > i \geq 0$, such that F'_j and F'_i are two faces of the same wall $M^{\mathfrak{b}}$ and such that all the chambers F_l, for $j-1 \geq l \geq i$, are on the same side of $M^{\mathfrak{b}}$.

If all the reflections s_{i_j} that compose the gallery of types $\tau_{\underline{i}}$ are distinct (including those given by the type t_P) then any gallery γ of $\Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, F_x)$ does not contain any buckle. Furthermore, any minimal gallery does not contain any buckle neither. Actually, if all the walls met by a gallery γ, $\mathcal{M}(\gamma) = \big\{M_{\beta_r}, ..., M_{\beta_1}, \{M_{x(\nu)}\}_{s_\nu \in t}\big\}$, are distinct, then γ does not contain any buckle.

Moreover, a buckle of a combinatorial gallery contains at least one bend (Definition 7).

Definition 9. *We say that a buckle $\mathfrak{b}$ of a gallery γ is maximal if all the buckles of γ on the wall $M^{\mathfrak{b}}$ are contained in $\mathfrak{b}$ (i.e. are sub-galleries of $\mathfrak{b}$).*

Now, we describe the set $\Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, F_x)$ using an operation analogous to the one that allowed us to describe the set $\Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, -)$ in the section 2.4.

Definition 10. *A part-folding $\overline{\phi}$ of a combinatorial gallery $\gamma \in \Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, F_x)$ which contains a buckle $\mathfrak{b}$ is the following operation. We take the part of γ determined by $\mathfrak{b}$ and we replace it by its image by the reflection $s_{M^{\mathfrak{b}}}$ associated to the wall $M^{\mathfrak{b}}$. We thus obtain a new combinatorial gallery $\overline{\phi}(\gamma) \in \Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, F_x)$.*

Further, we say that the part-folding $\overline{\phi}$ is towards F_B if all the chambers of the new buckle $s_{M^{\mathfrak{b}}}(\mathfrak{b})$ and F_B are on the same side of $M^{\mathfrak{b}}$.

The set $\Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, F_x)$ is invariant by part-foldings ($x = \overline{u}(P)$). Let us denote

$$\gamma_{u,w} \in \Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, F_x) \tag{1}$$

the unique gallery obtained from any other gallery of $\Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, F_x)$ by applying the maximum of part-foldings towards F_B. This gallery corresponds to the unique way to construct a reduced sub-decomposition of u from the chosen reduced decomposition of w by taking the reflections which define u the most possible towards the right.

So, we can recover all the combinatorial galleries above F_x by applying some part-foldings opposed to F_B (i.e. F_B does not belong to the image of the associated folding) to $\gamma_{u,w}$. For example, if $u = 1$, $\gamma_{1,w}$ is equal to $\gamma_0 = [1, ..., 1]$, the unique gallery of type $\tau_{\underline{i}}$ contained in the chamber F_B and all the galleries having $F_1 \subset F_B$ as target can be obtained from $\gamma_{1,w}$ thanks to part-foldings opposed to F_B.

The gallery $\gamma_{u,w}$ carries all the generator vectors of the combinatorial tangent space $T_x^c\overline{\Sigma}(B, \overline{w})$.

Proposition 5. *Still keeping the same notations,*

$$dim T_x^c\overline{\Sigma}(B, \overline{w}) \geq r = dim\overline{\Sigma}(B, \overline{w}).$$

Proof. Set $\gamma_{u,w} = [x_r, ..., x_1] \in P_{i_r} \times_B \cdots \times_B (P_{i_1}/B)$. Recall that the set of all the walls met by $\gamma_{u,w}$ is denoted $\mathcal{M}(\gamma_{u,w}) = \left\{M_{\beta_r}, ..., M_{\beta_1}, \{M_{u(\nu)}\}_{s_\nu \in t}\right\}$, where $\beta_j = x_r \cdots x_j(\alpha_{k_j})$ and $u = x_r \cdots x_1$.

Let us suppose that all the walls met by $\gamma_{u,w}$ are not distinct (otherwise $\Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, F_x) = \{\gamma_{u,w}\}$ and we have $dim T_x^c\overline{\Sigma}(B, \overline{w}) = r$).

On one side, let $\mathcal{T}(\gamma_{u,w}) \subset \{M_{\beta_r}, ..., M_{\beta_1}\}$ denote the set of walls crossed by $\gamma_{u,w}$, i.e. walls which separate F_B from F_x. Then $\#\mathcal{T}(\gamma_{u,w}) = l(u)$ and $\gamma_{u,w}$ never crosses again one of these walls, otherwise $\gamma_{u,w}$ would admits a buckle that we could fold towards F_B. Hence, from the remark 4, we obtain $l(u)$ generator vectors of $T_x^c\overline{\Sigma}(B, \overline{w})$. Furthermore, the roots associated to the walls which contain F_x cannot give any generator of the combinatorial tangent space. Hence, only the $m = l(w) - l(u)$ walls of $\mathcal{M}(\gamma_{u,w}) \setminus \mathcal{T}(\gamma_{u,w}) = \left\{M_{\beta_{i_m}}, ..., M_{\beta_{i_1}}\right\}$ may provide some new generators.

On the other side, the hypothesis done (i.e. all the walls met by $\gamma_{u,w}$ are not all distinct), implies that there exists only p walls, $\mathcal{D}(\gamma_{u,w}) = \left\{M_{\beta_{i'_p}}, ..., M_{\beta_{i'_1}}\right\} \subset$

$\left\{M_{\beta_{i_m}}, ..., M_{\beta_{i_1}}\right\}$ with $m > p \geq 1$, all distinct and also distinct from the walls crossed by $\gamma_{u,w}$ on which this gallery admits a generating bend. So, they give p more generator vectors of $T_x^c\overline{\Sigma}(B, \overline{w})$ (Remark 4).

The rest of the walls, $\mathcal{R}(\gamma_{u,w}) = \left\{M_{\beta_{i_m}}, ..., M_{\beta_{i_1}}\right\} \setminus \left\{M_{\beta_{i'_p}}, ..., M_{\beta_{i'_1}}\right\}$ are then equal to some crossed walls or some walls containing F_x or some walls of $\mathcal{D}(\gamma_{u,w})$. Hence, $\#\mathcal{R}(\gamma_{u,w}) = m - p = t + f + c$, where t is the number of walls of $\mathcal{R}(\gamma_{u,w})$ equal to crossed walls, f those equal to walls containing F_x and c those equal to walls of $\mathcal{D}(\gamma_{u,w})$. Moreover, $t = t_1 + \cdots + t_s$, $f = f_1 + \cdots + f_e$ and $c = c_1 + \cdots + c_b$, where the t_j's, f_j's and c_j's are respectively the numbers of walls of $\mathcal{R}(\gamma_{u,w})$ equal to a same wall of $\mathcal{T}(\gamma_{u,w})$, of $\{M_{u(\nu)}\}_{s_\nu \in t}$ and of $\mathcal{D}(\gamma_{u,w})$.

By the way, the gallery $\gamma_{u,w}$ admits s maximal buckles on crossed walls, e maximal buckles on walls containing F_x and b maximal buckles on walls of $\mathcal{D}(\gamma_{u,w})$ (cf. Definition 9). And each of these buckles provides, by applying part-foldings opposed to F_B, at least as much galleries containing a generating bend as the number t_j, f_j or c_j which corresponds to the maximal buckle we started with.

Indeed, let $\mathfrak{b}$ be one of these buckle, $\mathfrak{b}$ owns at least as much bends as the number t_j, f_j or c_j which corresponds to it. Let us apply to $\gamma_{u,w}$ the part-folding $\overline{\phi}$ fixed by $\mathfrak{b}$ (Definition 10), we obtain a new gallery $\overline{\phi}(\gamma_{u,w})$ which owns a buckle, image of $\mathfrak{b}$. Let (C', F') be a bend of this new buckle, then two cases may arise :

a) (C', F') is a generating bend on a wall that $\gamma_{u,w}$ did not meet ;

b) the gallery $\overline{\phi}(\gamma_{u,w})$ crosses the wall containing F', that means that it has a buckle on this wall, then we can apply a part-folding to it and repeat the discussion.

At the end, each maximal buckle will give at least as much generator vectors of $T_x^c\overline{\Sigma}(B, \overline{w})$ as the number t_j, f_j or c_j corresponding to it.

Hence, if we recount the number of generators which may arise from $\gamma_{u,w}$, we see that $dim T_x^c\overline{\Sigma}(\overline{w}) \geq r = l(w)$.

□

Remark 5. *All the combinatorial galleries above F_x are obtained by part-foldings opposed to F_B from $\gamma_{u,w}$. So, all the generators of $T_x^c\overline{\Sigma}(B, \overline{w})$ will be given using the process shown in the previous proof. Hence, the $T-$weights of this space, $\mathcal{R}_x$, will come from the walls we consider above.*

To close this section, we mention an important result of C. Contou Carrere.

Theorem 2 ([CC, Part II, §10]). *Let G be a $k-$semi-simple group of type A. For any parabolic subgroup $P \supset B$ and for all $T-$fixed point $x \in \overline{\Sigma}(B, \overline{w})$, the combinatorial tangent space $T_x^c\overline{\Sigma}(B, \overline{w})$ is equal to the Zariski tangent space $T_x\overline{\Sigma}(B, \overline{w})$.*

Thus, we can describe the singular locus of any Schubert variety in case of a group of type A.

4 Rational Smoothness of Schubert Varieties

First of all, we recall one of the definitions of rational smoothness [CP]. A projective variety Y of dimension r is rationally smooth at $y \in Y$ if there exists an neighborhood U of y such that the $l-$adic cohomology $H^i_{<y>}(U) = 0$ if $i \neq 2r$ and is one dimensional if $i = 2r$. The variety Y is rationally smooth if it is rationally smooth at every point.

A smooth point is rationally smooth, but there are rationally smooth Schubert varieties which are not smooth. This is the case for our example of the figure 3.2, see [C2, §7] for other examples.

4.1 Combinatorial Tangent Space and Rational Smoothness

Proposition 6. *Let $\overline{w} \in W/W_t$. For all $\overline{u} \leq \overline{w}$, we set $x = \overline{u}(P)$, then*

$$dim T^c_x \overline{\Sigma}(B, \overline{w}) = \#\left\{\beta \in R_+,\ \overline{s_\beta u} \leq \overline{w} \ \ \text{and}\ \overline{s_\beta u} \neq \overline{u}\right\}.$$

Proof. First, let $\beta \in R_+$ such that $\overline{s_\beta u} \leq \overline{w}$ and $\overline{s_\beta u} \neq \overline{u}$, we denote by M_β the wall associated to this root. Two cases may appear :

1) M_β separates C from F_x, the face associated to $\overline{u}(P)$. Then X_β is a generator vector of $T^c_x \overline{\Sigma}(B, \overline{w})$.

2) C and F_x are on the same side of M_β. Thus, the gallery $\gamma_{s_\beta u, w}$ (cf. (1) of the previous section for the definition) crosses this wall. Let us consider the gallery $\gamma \in \Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, F_x)$ image of the folding towards F_B associated to M_β of the gallery $\gamma_{s_\beta u, w}$. Then γ contains a generating bend on this wall and that gives a generator vector $X_{-\beta}$ of the combinatorial tangent space $T^c_x \overline{\Sigma}(B, \overline{w})$.

Moreover, the face F_x cannot belong to the wall M_β since this is equivalent to the fact that $\overline{s_\beta u} = \overline{u}$.

Second, let M_α be a wall that provides a generator vector of the combinatorial tangent space (i.e. X_α or $X_{-\alpha}$ belongs to $T^c_x \overline{\Sigma}(B, \overline{w})$). Let $\gamma \in \Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, F_x)$ denote a gallery which gives this generator. Then the gallery obtained from γ by keeping γ until the wall M_α and by following it with $s_\alpha(\gamma)$ is a combinatorial gallery that admits $F_{\overline{s_\alpha u}(P)}$ as a target. Hence, $\overline{s_\alpha u} \leq w$ and $\overline{s_\alpha u} \neq \overline{u}$ since a wall which gives a generator vector of $T^c_x \overline{\Sigma}(B, \overline{w})$ does not contain F_x.

These two constructions are inverse of each other, so they establish a bijection between the set of generator vectors of $T^c_x \overline{\Sigma}(B, \overline{w})$ and $\{\beta \in R_+,\ \overline{s_\beta u} \leq \overline{w}$ and $\overline{s_\beta u} \neq \overline{u}\}$, hence the proposition is proved.

□

Let $\overline{u} \leq \overline{w}$ inside W/W_t. Let us consider, following Carrell and Peterson ([CP]), the property $\mathbf{P}(\overline{u}, \overline{w})$ defined by :

$$\mathbf{P}(\overline{u}, \overline{w}) \Leftrightarrow \forall \overline{v},\ \overline{u} \leq \overline{v} \leq \overline{w},\ \#\left\{\beta \in R_+,\ \overline{u} < \overline{s_\beta v} \leq \overline{w} \ \text{and}\ \overline{s_\beta v} \neq \overline{v}\right\} = l(\overline{w}) - l(\overline{u}),$$

where the lengths are taken on the elements of minimal length of each class.

As it is remarked in [C1], this property is equivalent to the following, for all $\overline{u} \le \overline{v} \le \overline{w}$, $\#\{\beta \in R_+,\ \overline{s_\beta v} \le \overline{w}$ and $\overline{s_\beta v} \ne \overline{v}\} = l(\overline{w})$.

Now, let us suppose that $P = B$, i.e. we consider the Schubert varieties inside the variety of Borel subgroups. Then the last conditions, $\overline{s_\beta v} \ne \overline{v}$, in the above properties, are superfluous.

Let u, w be two elements of the weyl group such that $u \le w$. We denote by $P_{u,w}$ the Kashdan-Lusztig polynomial (cf. [KL], [D90]). We recall the following result due to Carrell and Peterson.

Theorem 3 ([CP, Theorem C]). *Let $u \le w$ inside the Weyl group W, then the property $\mathbf{P}(u, w)$ is equivalent to $P_{u,w} = 1$.*

But, in our case, the rational smoothness of a Schubert variety $\overline{\Sigma}(B, w)$ at the point $u(B)$ is characterized by the fact that $P_{u,w} = 1$ (see for exemple [KL] in the case the ground field has positive characteristic and [CP] for any algebraically closed field). Thus, as a corollary of what we said from the begining of this section (in particular, from Proposition 6), we get :

Theorem 4. *A Schubert variety $\overline{\Sigma}(B, w)$ is rationally smooth at $u(B)$, for $u \le w$, if and only if for all $u \le v \le w$, the dimension of the combinatorial tangent space $T^c_{v(B)}\overline{\Sigma}(B, w)$ is equal to $r = l(w)$.*

Furthermore, combining Theorem 2 of section 3.2 and Theorem 4, we have the following result, first obtained by Deodhar [D85] :

Corollary 1. *If G is of type A, then smoothness and rational smoothness for Schubert varieties in G/B are equivalent.*

Remark 6. *This characterization seems to go through in the parabolic case, that is for any $P \supset B$. But, in this case, it has to be taken into account the parabolic Kashdan-Lusztig polynomials to characterize the rational smoothness of Schubert varieties in G/P (cf. [D91, §6]).*

4.2 Combinatorial Tangent Space and $T-$invariant Curves

In this section, we keep the notations as in section 3 and we follow the definitions of Carrell and Peterson [CP] about $T-$invariant curves.

For any projective $T-$variety Y, a $T-$invariant curve in Y is the closure of a one dimensional orbit of T. Let us denote by $E(Y)$ the set of all the $T-$invariant curves in Y and if x is a $T-$fixed point of Y, $E(Y, x)$ denotes the set of all the $T-$invariant curves containing x.

First, we recall here some well-known properties of the $T-$invariant curves in

$$\overline{\Sigma}(B, \overline{w}) \hookrightarrow Par_{t_P}(G) \simeq G/P.$$

For a positive root $\alpha \in R^+$, we denote by U_α the unipotent subgroup of rank 1 of G associated to α. And $Z_\alpha =< U_\alpha, U_{-\alpha} >\subset G$, denotes the copy of Sl_2 associated to α.

Every $T-$invariant curve $\mathcal{C}$ in $Par_{t_P}(G)$ has the form $\mathcal{C} = Z_\alpha \overline{w}(P)$ for some $\overline{w} \in W/W_{t_P}$ and $\alpha \in R^+$ [CP, Theorem F].

Let $\overline{u} \in W/W_{t_P}$ such that $\overline{u} \leq \overline{w}$ for the relative Chevalley-Bruhat order (defined on the set of minimal length coset representatives for W/W_{t_P}). Set $x = \overline{u}(P)$, the $T-$fixed point corresponding to $\overline{u}$.

Carrell and Peterson proved that $Z_\alpha \overline{u}(P) \in E\big(\overline{\Sigma}(\overline{w}, B), x\big)$ if and only if $\overline{u} \leq \overline{w}$ and $\overline{s_\alpha u} \leq \overline{w}$ (and $\overline{s_\alpha u} \neq \overline{u}$) [CP, Theorem F].

Thus, from the proposition 6 (more precisely from the proof of this proposition), we obtain,

Proposition 7. *To each generator vector of the combinatorial tangent space $T^c_x\overline{\Sigma}(B,\overline{w})$ corresponds a $T-$invariant curve in $\overline{\Sigma}(\overline{w})$ which contains x and this space is spanned by the tangent vectors to each of these curves. Hence, $\#E\big(\overline{\Sigma}(\overline{w}), x\big) = dim T^c_x\overline{\Sigma}(B,\overline{w})$.*

Now, we go to the $T-$invariant curves in the Bott-Samelson variety $\hat{\Sigma}(B, \tau_{\underline{i}})$.

The set of $T-$fixed points of this variety $\hat{\Sigma}(B, \tau_{\underline{i}})^T$ is finite and is equal to the finite set $\Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, -)$ of all the combinatorial galleries of $\hat{\Sigma}(B, \tau_{\underline{i}})$. Hence, there are at least r $T-$invariant curves passing through each combinatorial gallery [CP, §2].

Definition 11. *A $T-$invariant curve $\mathfrak{C}$ of the Bott-Samelson variety will be called a combinatorial $T-$invariant curve if it contains two combinatorial galleries $\gamma = [\gamma_r, ..., \gamma_1]$ and $\delta = [\delta_r, ..., \delta_1]$ for which there exists a unique $j_0 \in \{r, ..., 1\}$ such that for all $j \neq j_0$, $\gamma_j = \delta_j$ and $\gamma_{j_0} = s_{\alpha_{k_{j_0}}}\delta_{j_0}$.*

Such a curve $\mathfrak{C}$ maps by π on a $T-$invariant curve in the Scubert variety, more precisely $\pi(\mathfrak{C}) = Z_\beta\pi(\gamma) = Z_\beta\pi(\delta)$, where $\beta = \gamma_r \cdots \gamma_{j_0+1}(\alpha_{k_{j_0}})$. However, there exists some $T-$invariant curves in the Bott-Samelson variety that are not of this kind. This is due to the fact that the action of the torus T on $\hat{\Sigma}(B, \tau_{\underline{i}})$ is not "special" in the sense of Carrell and Peterson [CP, §2].

So, we use the notation $\mathfrak{C} = (\gamma \frown \delta)$ to describe such a $T-$invariant curve, which contains γ and δ as two distinct $T-$fixed points.

Moreover, some of the galleries of $\Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, -)$ can be obtained from the gallery $\gamma_w \in \Gamma(\mathbb{A}, \tau_{\underline{i}}, F_B, F_{\overline{w}(P)})$ by a finite number of foldings towards F_B (cf. 2.4, Proposition 1). Hence, if $\mathfrak{C} = (\gamma \frown \delta) \in E\big(\hat{\Sigma}(B, \tau_{\underline{i}})\big)$, where δ is obtained from γ by a folding ϕ_α towards F_B ($\alpha \in R^+$), we will denote it by $\mathfrak{C}_\alpha = (\gamma \overset{\phi_\alpha}{\frown} \delta)$. Furthermore, $\pi(\mathfrak{C}_\alpha) = Z_\alpha\pi(\gamma) = Z_\alpha\pi(\delta)$.

We now define the finite graph $\mathcal{BS}^c_T$ of the combinatorial $T-$invariant curves in $\hat{\Sigma}(B, \tau_{\underline{i}})$.

The set of vertices $\mathcal{B}S^c_T$ is defined as the set $\hat{\Sigma}(B,\tau_{\underline{i}})^T$ of all $T-$fixed points in the Bott-Samelson variety, i.e. the combinatorial galleries $\Gamma(\mathbb{A},\tau_{\underline{i}},F_B,-)$.

And the set of edges $\mathcal{B}S^c_T$ is defined as the set of combinatorial curves in $E\big(\hat{\Sigma}(B,\tau_{\underline{i}})\big)$. For γ and δ in $\Gamma(\mathbb{A},\tau_{\underline{i}},F_B,-)$, there is an edge between them if $\mathfrak{C}=(\gamma\frown\delta)$ is a combinatorial $T-$invariant curve in $\hat{\Sigma}(B,\tau_{\underline{i}})$. If this curve is associated to a folding ϕ_α towards F_B ($\alpha>0$), we label the corresponding edge of $\mathcal{B}S^c_T$ by ϕ_α.

Moreover, we form into rows the vertices of the graph (i.e. the combinatorial galleries) depending on the distance that separate the chamber F_B from the target of the gallery. Thus the gallery γ_w will be the unique vertex at the bottom row of the graph and $\gamma_{1,w}$ will belong to the top row. As this distance is exactly the length of the element of the Weyl group that gives the target face of the gallery, we have a surjective morphism of graphs from $\mathcal{B}S^c_T$ to the Bruhat subgraph of (W,S) associated to w, hence we say that $\mathcal{B}S^c_T$ is above the Bruhat graph of w.

Furthermore, from the previous definition, exactly $r=dim\hat{\Sigma}(B,\tau_{\underline{i}})$ edges meet at each vertex.

To see what such a graph looks like, let us take the example of §3.2 up again. We recall that the group G is of type B_2 and we fix $P=B$ and $w=s_\alpha s_\beta s_\alpha$. The set of combinatorial galleries $\Gamma(\mathbb{A},\tau_{\underline{i}},F_B,-)$ is represented in the figure 3.2. And from their expressions in terms of generators of the Weyl group as points of $P_\alpha\times_B P_\beta\times_B P_\alpha/B$, it is easy to describe the combinatorial $T-$invariant curves between them. So in this case the graph $\mathcal{B}S_T$ has the following form.

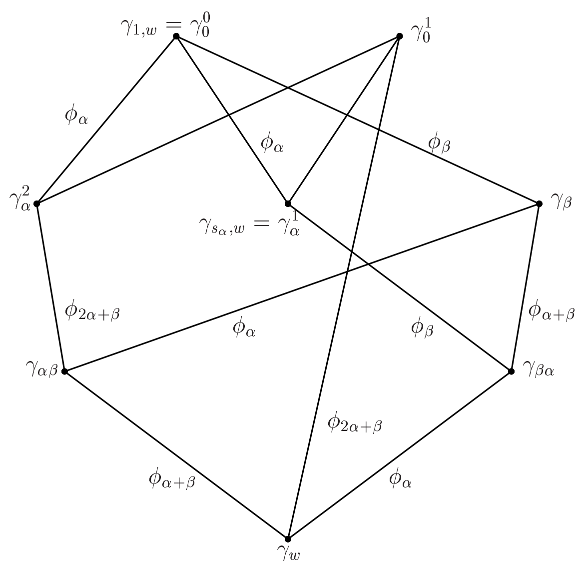

Figure 4.2. Graph of combinatorial $T-$invariant curves of $\hat{\Sigma}(B,\tau_{\underline{i}})$, case B_2, $w=s_\alpha s_\beta s_\alpha$.

As we can see on this example, some of the $T-$invariant curves of $\hat{\Sigma}(B,\tau_{\underline{i}})$ are

not related to a folding. For the others, the folding can be easily read on the figure 3.2.

Now, we return to the general case and we describe a way to read off the generators of the combinatorial tangent space from the graph $\mathcal{B}S_T^c$.

First, let $x = \overline{u}(P)$ be a $T-$fixed point of $\overline{\Sigma}(B, \overline{w})$, i.e. a vertex of the Bruhat graph of w.

Let us consider all the vertices above x, that is, all the combinatorial galleries whose target is F_x. We denote by $\mathcal{R}_x^+$ (resp. $\mathcal{R}_x^-$) the set of all the distinct roots that, eventually, label the edges leaving to the top (resp. arriving from the bottom to) these vertices. Then, we deduce from what has been said from the begining of this section the following proposition.

Proposition 8. *The set $\mathcal{R}_x$ of all the $T-$weights of the combinatorial tangent space $T_x^c\overline{\Sigma}(B, \overline{w})$ (cf. Section 3.2) actually equals $\mathcal{R}_x^+ \amalg (-\mathcal{R}_x^-)$.*

This proposition shows up a new way to calculate the generator vectors of the combinatorial tangent space. For instance, if we go back to the previous example, it is easy to see on the graph that $T^c_{s_\alpha(B)}\overline{\Sigma}(B, w)$ admits the basis $\{X_{-\beta}, X_{-2\alpha-\beta}, X_\alpha\}$ and if $x = s_\alpha s_\beta(B) = s_\alpha s_\beta B(s_\alpha s_\beta)^{-1}$, then $T_x^c\overline{\Sigma}(B, w) =< X_\alpha, X_{2\alpha+\beta}, X_{-\alpha-\beta} >$. Thus, we recover the results of section 3.2.

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